Zakharov-Shabat system and hyperbolic pseudoanalytic function theory

Viktor G. Kravchenko*, Vladislav V. Kravchenko† and Sébastien Tremblay‡

*Departamento de Matemática, F.C.T., Universidade do Algarve, Campus de Gambelas, 8000-810 Faro, Portugal
†Departamento de Matemática, CINVESTAV del IPN, Unidad Querétaro, Libramiento Norponiente No. 2000 C.P. 76230 Fracc. Real de Juriquilla, Querétaro, Mexico
‡Département de mathématiques et d'informatique, Université du Québec, Trois-Rivières, Québec, G9A 5H7, Canada

Abstract

In [1] a hyperbolic analogue of pseudoanalytic function theory was developed. In the present contribution we show that one of the central objects of the inverse problem method the Zakharov-Shabat system is closely related to a hyperbolic Vekua equation for which among other results a generating sequence and hence a complete system of formal powers can be constructed explicitly.

1 Introduction

In [1] a hyperbolic analogue of pseudoanalytic function theory was developed using the algebra of hyperbolic numbers instead of complex numbers in the case of Bers [2, 3, 4]. Hyperbolic numbers $\mathbb{D}$, also called duplex numbers, is a commutative ring with zero divisors defined in the plane by $\mathbb{D} = \{z = x + \mathrm{j}t : x, t \in \mathbb{R},\ \mathrm{j}^2 = 1\}$ (see [5, 6, 7] for instance). With the aid of the hyperbolic pseudoanalytic function theory in [1] a procedure for constructing an infinite system of solutions for the Klein-Gordon equation $(\square - \nu(x,t))\varphi(x,t) = 0$ was introduced. This system is a hyperbolic analogue of formal powers in the sense of Bers. In the present paper we consider the Zakharov-Shabat system and show that it is closely related to a hyperbolic Vekua equation. Using the above mentioned procedure an infinite system of its solutions is obtained.

2 Hyperbolic pseudoanalytic functions

In this section we present some results of hyperbolic pseudoanalytic function theory. For more details see [1].

We will consider the variable $z = x + t\mathrm{j}$, where x and t are real variables and the corresponding formal differential operators

$$\partial_z = \frac{1}{2}\left(\partial_x + \mathrm{j}\partial_t\right) \text{ and } \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x - \mathrm{j}\partial_t\right).$$

Notation $f_{\bar{z}}$ or f_z means the application of $\partial_{\bar{z}}$ or ∂_z respectively to a hyperbolic function $f(z) = u(z) + v(z)\mathrm{j}$. We have the following result in the hyperbolic function theory.

Lemma 1 *Let $f(x + t\mathrm{j}) = u(x,t) + v(x,t)\mathrm{j}$ be a hyperbolic function where u_x, u_t, v_x and v_t exist, and are continuous in a neighborhood of z_0. The derivative*

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ (z - z_0 \ inv.)}} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, if and only if

$$f_{\bar{z}}(z_0) = 0.$$

Moreover, $f'(z_0) = f_z(z_0)$ and $f'(z_0)$ is invertible if and only if $\det \mathcal{J}_f(z_0) \neq 0$.

The hyperbolic pseudoanalytic function theory is based on assigning the part played by 1 and j in an arbitrary hyperbolic function $f = u(x,t)1 + v(x,t)\mathrm{j}$ to two essentially arbitrary hyperbolic functions F and G. We assume that these functions are defined and twice continuously differentiable in some open domain $\Omega \subset \mathbb{D}$. We require that

$$\mathrm{Im}\{\overline{F(z)}G(z)\} \neq 0.$$

Under this condition, (F, G) will be called a "generating pair" in Ω. Notice that $\mathrm{Im}\{\overline{F(z)}G(z)\} = \begin{vmatrix} \mathrm{Re}\{F(z)\} & \mathrm{Re}\{G(z)\} \\ \mathrm{Im}\{F(z)\} & \mathrm{Im}\{G(z)\} \end{vmatrix}$. It follows, from Cramer's theorem, that for every z_0 in Ω we can find unique constants $\lambda_0, \mu_0 \in \mathbb{R}$ such that $w(z_0) = \lambda_0 F(z_0) + \mu_0 G(z_0)$. More generally we have the following result.

Theorem 2 *Let (F, G) be generating pair in some open domain Ω. If $w(z) : \Omega \subset \mathbb{D} \to \mathbb{D}$, then there exist **unique** functions $\phi(z), \psi(z) : \Omega \subset \mathbb{D} \to \mathbb{R}$ such that*

$$w(z) = \phi(z)F(z) + \psi(z)G(z), \ \forall z \in \Omega.$$

Moreover, we have the following explicit formulas for ϕ and ψ:

$$\phi(z) = \frac{\mathrm{Im}[\overline{w(z)}G(z)]}{\mathrm{Im}[\overline{F(z)}G(z)]}, \ \psi(z) = -\frac{\mathrm{Im}[\overline{w(z)}F(z)]}{\mathrm{Im}[\overline{F(z)}G(z)]}.$$

Consequently, every hyperbolic function w defined in some subdomain of Ω admits the unique representation $w = \phi F + \psi G$ where the functions ϕ and ψ are real valued. Thus, the pair (F, G) generalizes the pair $(1, \mathrm{j})$ which corresponds to hyperbolic analytic function theory.

We say that $w : \Omega \subset \mathbb{D} \to \mathbb{D}$ possesses at z_0 the (F, G)-derivative $\dot{w}(z_0)$ if the (finite) limit

$$\dot{w}(z_0) = \lim_{\substack{z \to z_0 \\ (z - z_0 \ inv.)}} \frac{w(z) - \lambda_0 F(z) - \mu_0 G(z)}{z - z_0}$$

exists.

The following expressions are called the characteristic coefficients of the pair (F, G):

$$a_{(F,G)} = -\frac{\bar{F}G_{\bar{z}} - F_{\bar{z}}\bar{G}}{F\overline{G} - \overline{F}G}, \quad b_{(F,G)} = \frac{FG_{\bar{z}} - F_{\bar{z}}G}{F\overline{G} - \overline{F}G}$$
$$A_{(F,G)} = -\frac{\overline{F}G_z - F_z\overline{G}}{F\overline{G} - \overline{F}G}, \quad B_{(F,G)} = \frac{FG_z - F_zG}{F\overline{G} - \overline{F}G}.$$

Theorem 3 *Let (F,G) be a generating pair in some open domain Ω. Every hyperbolic function $w \in C^1(\Omega)$ admits the unique representation $w = \phi F + \psi G$ where $\phi, \psi : \Omega \subset \mathbb{D} \to \mathbb{R}$. Moreover, the (F, G)-derivative $\dot{w} = \frac{\mathrm{d}_{(F,G)} w}{\mathrm{d}z}$ of $w(z)$ exists and has the form*

$$\dot{w} = \phi_z F + \psi_z G = w_z - A_{(F,G)} w - B_{(F,G)} \overline{w} \tag{1}$$

if and only if

$$w_{\bar{z}} = a_{(F,G)} w + b_{(F,G)} \overline{w}. \tag{2}$$

The equation (2) can be rewritten in the following form

$$\phi_{\bar{z}} F + \psi_{\bar{z}} G = 0. \tag{3}$$

Equation (2) is called “hyperbolic Vekua equation” and any continuously differentiable solutions of this equation are called “hyperbolic (F, G)-pseudoanalytic functions”.

Remark 4 *The functions F and G are hyperbolic (F, G)-pseudoanalytic, and $\dot{F} \equiv \dot{G} \equiv 0$.*

Definition 5 *Let (F, G) and (F_1, G_1) - be two generating pairs in Ω. (F_1, G_1) is called successor of (F, G) and (F, G) is called predecessor of (F_1, G_1) if*

$$a_{(F_1,G_1)} = a_{(F,G)} \qquad \textit{and} \qquad b_{(F_1,G_1)} = -B_{(F,G)}.$$

The importance of this definition becomes obvious from the following statement.

Theorem 6 *Let w be a hyperbolic (F, G)-pseudoanalytic function and let (F_1, G_1) be a successor of (F, G). If $\dot{w} = W \in C^1(\Omega)$ then W is a hyperbolic (F_1, G_1)-pseudoanalytic function.*

Definition 7 *Let (F,G) be a generating pair. Its adjoint generating pair $(F,G)^* = (F^*,G^*)$ is defined by the formulas*

$$F^* = -\frac{2\overline{F}}{F\overline{G}-\overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G}-\overline{F}G}.$$

The (F,G)-integral is defined as follows

$$\int_\Gamma w\,\mathrm{d}_{(F,G)}z = F(z_1)\mathrm{Re}\int_\Gamma G^* w\,\mathrm{d}z + G(z_1)\mathrm{Re}\int_\Gamma F^* w\,\mathrm{d}z$$

where Γ is a rectifiable curve leading from z_0 to z_1.

If $w = \phi F + \psi G$ is a hyperbolic (F,G)-pseudoanalytic function where ϕ and ψ are real valued functions then

$$\int_{z_0}^{z} \dot{w}\,\mathrm{d}_{(F,G)}\zeta = w(z) - \phi(z_0)F(z) - \psi(z_0)G(z). \tag{4}$$

This integral is path-independent and represents the (F,G)-antiderivative of $\dot{w}$. The expression $\phi(z_0)F(z) + \psi(z_0)G(z)$ in (4) can be seen as a “pseudoanalytic constant” of the generating pair (F,G) in Ω.

A continuous function $W(z)$ defined in a domain Ω will be called (F,G)-integrable if for every closed curve Γ situated in a simply connected subdomain of Ω the following equality holds

$$\oint_\Gamma W\,\mathrm{d}_{(F,G)}z = 0.$$

Theorem 8 *Let W be a hyperbolic (F,G)-pseudoanalytic function. Then W is (F,G)-integrable.*

Definition 9 *A sequence of generating pairs $\{(F_m,G_m)\}$ with $m\in\mathbb{Z}$, is called a generating sequence if (F_{m+1},G_{m+1}) is a successor of (F_m,G_m). If $(F_0,G_0) = (F,G)$, we say that (F,G) is embedded in $\{(F_m,G_m)\}$.*

Definition 10 *A generating sequence $\{(F_m,G_m)\}$ is said to have period $\mu > 0$ if $(F_{m+\mu},G_{m+\mu})$ is equivalent to (F_m,G_m) that is their characteristic coefficients coincide.*

Let w be a hyperbolic (F,G)-pseudoanalytic function. Using a generating sequence in which (F,G) is embedded we can define the higher derivatives of w by the recursion formula

$$w^{[0]} = w; \qquad w^{[m+1]} = \frac{\mathrm{d}_{(F_m,G_m)}w^{[m]}}{\mathrm{d}z}, \quad m = 1,2,\ldots$$

Definition 11 *The formal power $Z_m^{(0)}(a, z_0; z)$ with center at $z_0 \in \Omega$, coefficient a and exponent 0 is defined as the linear combination of the generators F_m, G_m with real constant coefficients λ, μ chosen so that $\lambda F_m(z_0) + \mu G_m(z_0) = a$. The formal powers with exponents $n = 1, 2, \ldots$ are defined by the recursion formula*

$$Z_m^{(n)}(a, z_0; z) = n \int_{z_0}^{z} Z_{m+1}^{(n-1)}(a, z_0; \zeta) \mathrm{d}_{(F_m, G_m)} \zeta. \tag{5}$$

This definition implies the following properties.

(i) $Z_m^{(n)}(a, z_0; z)$ is a (F_m, G_m)- hyperbolic pseudoanalytic function of z.

(ii) If a_1 and a_2 are real constants, then $Z_m^{(n)}(a_1 + \mathrm{j}a_2, z_0; z) = a_1 Z_m^{(n)}(1, z_0; z) + a_2 Z_m^{(n)}(\mathrm{j}, z_0; z)$.

(iii) The formal powers satisfy the differential relations

$$\frac{\mathrm{d}_{(F_m, G_m)} Z_m^{(n)}(a, z_0; z)}{dz} = n Z_{m+1}^{(n-1)}(a, z_0; z).$$

(iv) The asymptotic formulas

$$Z_m^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \quad z \to z_0$$

hold.

3 Zakharov-Shabat system and a hyperbolic Vekua equation

Inverse scattering problems involving coupling mode have been investigated by many authors. When the medium concerned is treated as a continuously varying one, the one-dimensional case is usually associated with Zakharov-Shabat coupling-mode equations (see, e.g., [8])

$$\partial_x n_1 + ikn_1 = s(x) n_2, \qquad \partial_x n_2 - ikn_2 = -s(x) n_1, \tag{6}$$

where the functions n_1, n_2 (the modes) and the potential $s(x)$ are complex valued functions and the parameter k (the wave number) is complex. This system is frequently considered as a Fourier transform of the following system

$$\partial_x n_+ + \partial_t n_+ = s(x) n_-, \qquad \partial_x n_- - \partial_t n_- = -s(x) n_+. \tag{7}$$

Consider the following functions

$$u = n_- + n_+, \qquad v = n_- - n_+.$$

We have

$$\partial_x u - \partial_t v = sv, \qquad \partial_x v - \partial_t u = -su.$$

This system can be written in the form

$$W_{\overline{z}} = -\frac{s(x)\mathrm{j}}{2}\overline{W} \tag{8}$$

where $z = x + \mathrm{j}t$ ($\mathrm{j}^2 = 1$), $W = u + \mathrm{j}v$, $W_{\overline{z}} = \frac{1}{2}(\partial_x - \mathrm{j}\partial_t)W$.

The coefficient in this hyperbolic Vekua equation in general is not representable in the form of a logarithmic derivative of a scalar function (see [1]). Nevertheless we are able to construct a corresponding generating pair:

$$F(x) = \cos S(x) - \mathrm{j}\sin S(x), \qquad G(x) = \sin S(x) + \mathrm{j}\cos S(x),$$

where S is an antiderivative of s. Notice that $\mathrm{Im}(\overline{F}G) \equiv 1$.

In order to introduce the (F,G)-derivative in the sense of Bers let us calculate the characteristic coefficients $A_{(F,G)}$ and $B_{(F,G)}$. For this the following auxiliary formulae are helpful

$$F_{\overline{z}} = F_z = -\frac{s}{2}G \qquad \text{and} \qquad G_{\overline{z}} = G_z = \frac{s}{2}F.$$

Then

$$A_{(F,G)} = 0 \qquad \text{and} \qquad B_{(F,G)} = -\frac{s(x)\mathrm{j}}{2}$$

(we used the relations $F\overline{F} + G\overline{G} = 0$ and $F^2 + G^2 = 2$). Thus, the (F,G)-derivative of solutions of (8) has the form

$$w = \dot{W} = W_z + \frac{s(x)\mathrm{j}}{2}\overline{W}$$

and is a solution of the equation

$$w_{\overline{z}} = \frac{s(x)\mathrm{j}}{2}\overline{w}$$

for which a generating pair can be constructed as well

$$F_1(x) = \cos S(x) + \mathrm{j}\sin S(x), \qquad G_1(x) = -\sin S(x) + \mathrm{j}\cos S(x).$$

The generating sequence $\{(F_m, G_m)\}$ has then the form

$$F_m = \cos S(x) + (-1)^{m+1}\mathrm{j}\sin S(x), \quad G_m = (-1)^m \sin S(x) + \mathrm{j}\cos S(x),$$

with

$$\left(W^{[n]}\right)_{\overline{z}} = (-1)^{n+1}\frac{s(x)\mathrm{j}}{2}\overline{W^{[n]}} \quad \Leftrightarrow \quad W^{[n+1]} = \left(W^{[n]}\right)_z + (-1)^n\frac{s(x)\mathrm{j}}{2}\overline{W^{[n]}}.$$

That is, it is periodic with a period 2. In this case the whole system of formal powers can be constructed explicitly. We find that

$$F_m^* = G_m, \qquad G_m^* = F_m.$$

Let us now construct the formal powers of (8) on the *time-like* subdomain $\Omega = \{z = x + \mathrm{j}t \mid 0 < x < t < \infty\}$ of the hyperbolic plane. For $a, z_0 \in \Omega$ with $a = a_1 + \mathrm{j}a_2$ and $z_0 = x_0 + \mathrm{j}t_0$, we have by definition $Z^{(0)}(a, z_0; z_0) = \lambda F(z_0) + \mu G(z_0) = a_1 + \mathrm{j}a_2$, where $\lambda, \mu \in \mathbb{R}$. We solve easily the system of two linear equations and obtain $\lambda = a_1\alpha - a_2\beta$ and $\mu = a_1\beta + a_2\alpha$, where we defined $\alpha = \cos S(x_0)$ and $\beta = \sin S(x_0)$. Therefore, we obtain

$$Z^{(0)}(a, z_0; z) = (a_1\alpha - a_2\beta)F(z) + (a_1\beta + a_2\alpha)G(z).$$

Let us now calculate $Z^{(1)}(a, z_0; z)$. For that we need to calculate $Z_1^{(0)}(a, z_0; z)$ before. We have $Z_1^{(0)}(a, z_0; z_0) = \lambda F_1(z_0) + \mu G_1(z_0) = a_1 + \mathrm{j}a_2$, where $\lambda, \mu \in \mathbb{R}$. Again we solve easily the linear system and find $Z_1^{(0)}(a, z_0; z) = (a_1\alpha + a_2\beta)F_1(z) + (-a_1\beta + a_2\alpha)G_1(z)$. Therefore, we obtain

$$\begin{aligned} Z^{(1)}(a, z_0; z) &= \int_{z_0}^{z} Z_1^{(0)}(a, z_0; \zeta)\mathrm{d}_{(F,G)}\zeta \\ &= F(z)\Big[(a_1\alpha + a_2\beta)X^{(1)} + (a_1\beta - a_2\alpha)Y^{(1)} + (t - t_0)(-a_1\beta + a_2\alpha)\Big] \\ &+ G(z)\Big[(-a_1\beta + a_2\alpha)X^{(1)} + (a_1\alpha + a_2\beta)Y^{(1)} + (t - t_0)(a_1\alpha + a_2\beta)\Big], \end{aligned}$$

where $X^{(1)} = X^{(1)}(x_0; x)$ and $Y^{(1)} = Y^{(1)}(x_0; x)$ are defined by

$$X^{(1)}(x_0; x) := \int_{x_0}^{x} \cos\big(2S(\xi)\big)\mathrm{d}\xi \quad \text{and} \quad Y^{(1)}(x_0; x) := \int_{x_0}^{x} \sin\big(2S(\xi)\big)\mathrm{d}\xi. \quad (9)$$

Now, if we want to find $Z^{(2)}(a, z_0; z)$ we need to calculate first $Z_1^{(1)}(a, z_0; z)$; which is themselves obtained from $Z_2^{(0)}(a, z_0; z)$. However, since the generating pairs are of period 2 we have that $Z_2^{(0)}(a, z_0; z) = Z^{(0)}(a, z_0; z)$. Then we obtain

$$\begin{aligned} Z_1^{(1)}(a, z_0; z) &= \int_{z_0}^{z} Z^{(0)}(a, z_0; \zeta)\mathrm{d}_{(F_1,G_1)}\zeta \\ &= F_1(z)\Big[(a_1\alpha - a_2\beta)X^{(1)} + (a_1\beta + a_2\alpha)Y^{(1)} + (t - t_0)(a_1\beta + a_2\alpha)\Big] \\ &\quad + G_1(z)\Big[(a_1\beta + a_2\alpha)X^{(1)} + (-a_1\alpha + a_2\beta)Y^{(1)} + (t - t_0)(a_1\alpha - a_2\beta)\Big]. \end{aligned}$$

We are now able to calculate $Z^{(2)}(a, z_0; z)$:

$$\begin{aligned}
Z^{(2)}(a, z_0; z) &= 2\int_{z_0}^{z} Z_1^{(1)}(a, z_0; \zeta)\mathrm{d}_{(F,G)}\zeta \\
&= F(z)\Big[(a_1\alpha - a_2\beta)X^{(2)} + (a_1\beta + a_2\alpha)\widetilde{X}^{(2)} + 2(t-t_0)(a_1\beta + a_2\alpha)X^{(1)} \\
&\quad +(-a_1\beta - a_2\alpha)\widetilde{Y}^{(2)} + (a_1\alpha - a_2\beta)Y^{(2)} + 2(t-t_0)(-a_1\alpha + a_2\beta)Y^{(1)} \\
&\quad +\tfrac{t-t_0}{x-x_0}(a_1\beta + a_2\alpha)I^{(2)} + \tfrac{t-t_0}{x-x_0}(-a_1\alpha + a_2\beta)\widetilde{I}^{(2)} + 2(t-t_0)^2(a_1\alpha - a_2\beta)\Big] \\
&\quad G(z)\Big[(a_1\beta + a_2\alpha)X^{(2)} + (-a_1\alpha + a_2\beta)\widetilde{X}^{(2)} + 2(t-t_0)(a_1\alpha - a_2\beta)X^{(1)} \\
&\quad +(a_1\alpha - a_2\beta)\widetilde{Y}^{(2)} + (a_1\beta + a_2\alpha)Y^{(2)} + 2(t-t_0)(a_1\beta + a_2\alpha)Y^{(1)} \\
&\quad +\tfrac{t-t_0}{x-x_0}(a_1\alpha - a_2\beta)I^{(2)} + \tfrac{t-t_0}{x-x_0}(a_1\beta + a_2\alpha)\widetilde{I}^{(2)} + 2(t-t_0)^2(a_1\beta + a_2\alpha)\Big]
\end{aligned}$$

where the functions $X^{(n)}$, $Y^{(n)}$, $\widetilde{X}^{(n)}$, $\widetilde{Y}^{(n)}$, $I^{(n)}$ and $\widetilde{I}^{(n)}$, depending on x_0 and x, are given by

$$\begin{aligned}
&X^{(n)}(x_0; x) := n\int_{x_0}^{x} X^{(n-1)}(x_0; \xi)\ \cos\big(2S(\xi)\big)\mathrm{d}\xi,\ Y^{(n)}(x_0; x) := n\int_{x_0}^{x} Y^{(n-1)}(x_0; \xi)\ \sin\big(2S(\xi)\big)\mathrm{d}\xi, \\
&\widetilde{X}^{(n)}(x_0; x) := n\int_{x_0}^{x} Y^{(n-1)}(x_0; \xi)\ \cos\big(2S(\xi)\big)\mathrm{d}\xi,\ \widetilde{Y}^{(n)}(x_0; x) := n\int_{x_0}^{x} X^{(n-1)}(x_0; \xi)\ \sin\big(2S(\xi)\big)\mathrm{d}\xi, \\
&I^{(n)}(x_0; x) := n\int_{x_0}^{x} X^{(n-1)}(x_0; \xi)\mathrm{d}\xi,\ \widetilde{I}^{(n)}(x_0; x) := n\int_{x_0}^{x} Y^{(n-1)}(x_0; \xi)\mathrm{d}\xi,
\end{aligned}$$

with $X^{(0)}(x_0, x) = Y^{(0)}(x_0, x) = \widetilde{X}^{(0)}(x_0, x) = \widetilde{Y}^{(0)}(x_0, x) = 1$.

4 Conclusion

We have shown that the Zakharov-Shabat system is related to a Vekua equation for which a generating sequence is found. Using the generating sequence it is possible to obtain the associated formal powers which are solutions of the given Vekua equation. Therefore, we obtain an infinite set of solutions for the Zakharov-Shabat system.

Acknowledgments

The research of V. G. Kravchenko was partially supported by Centro de Análise Funcional e Aplicações Aplicações do Instituto Superior Técnico (Portugal). V. V. Kravchenko wishes to express his gratitude to CONACYT (Mexico) for supporting this work via the research project 50424. The research of S. Tremblay was supported in part by grant from CRSNG of Canada.

References

[1] V. V. Kravchenko, D. Rochon and S. Tremblay, *J. Phys. A: Math. Theor.* **41**, 65205 (2008).

[2] L. Bers, *Theory of Pseudo-Analytic Functions*, New York University, New York, 1952.

[3] S. Agmon and L. Bers, *Proc. Amer. Math. Soc.* **3**, 757 (1952).

[4] L. Bers, *Bull. Am. Math. Soc.* **62**, 291 (1956).

[5] G. Sobczyk, *Coll. Maths. Jour.* **26**, 268 (1995).

[6] A. F. Motter and M. A. F. Rosa, *Adv. Appl. Cliff. Alg.* **8**, 109 (1998).

[7] Guo Chun Wen, *Linear and Quasilinear Complex Equations of Hyperbolic and Mixed Type*, Taylor & Francis, London, 2003.

[8] G. L. Lamb, *Elements of soliton theory*, Dover, 1995.

[9] I. N. Vekua, *Generalized Analytic Functions*, Pergamon Press Ltd, Oxford, 1962.